01/04/2025

13.29

$$\lim_{x\to 0}\frac{f(x)-x}{x^2}=2005$$

α) i) Έστω $g(x)=\frac{f(x)-x}{x^2}$, $x\in\mathbb{R}^*$ με $\lim_{x\to 0} g(x)=2005$.

Κοντά στο 0 είναι $f(x)=x^2g(x)+x$ άρα και $\lim_{x\to 0} f(x)=\lim_{x\to 0}(x^2g(x)+x)=$
$=0\cdot 2005+0=0$. Η f είναι συνεχής άρα και στο 0 οπότε $f(0)=\lim_{x\to 0} f(x)=0$.

ii) $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^2g(x)+x}{x}=\lim_{x\to 0}(xg(x)+1)=0\cdot 2005+1=1\in\mathbb{R}$,

άρα ... $f'(0)=1$

β) $\lim_{x\to 0}\frac{x^2+\lambda(f(x))^2}{2x^2+(f(x))^2}=3 \Leftrightarrow \lim_{x\to 0}\frac{1+\lambda\cdot\left(\frac{f(x)}{x}\right)^2}{2+\left(\frac{f(x)}{x}\right)^2}=3 \Leftrightarrow$

$\Leftrightarrow \frac{1+\lambda}{2+1}=3 \Leftrightarrow 1+\lambda=9 \Leftrightarrow \boxed{\lambda=8}$.

γ) $f'(x)>f(x)\ \forall x\in\mathbb{R}$. (1)

i) $xf(x)>0\ \forall x\neq 0$

$(1)\Leftrightarrow f'(x)-f(x)>0 \overset{e^{-x}>0}{\Longleftrightarrow} e^{-x}f'(x)-e^{-x}f(x)>0 \Leftrightarrow (\underbrace{e^{-x}f(x)}_{\phi(x)})'>0\ \forall x\in\mathbb{R}$.

$\phi'(x)>0\ \forall x\in\mathbb{R} \Longrightarrow \phi:\uparrow\mathbb{R}$.

- $x>0 \overset{\phi:\uparrow}{\Longleftrightarrow} \phi(x)>\phi(0) \Leftrightarrow e^{-x}f(x)>0 \overset{e^{-x}>0}{\Longleftrightarrow} f(x)>0$
- $x<0 \overset{\phi:\uparrow}{\Longleftrightarrow} \phi(x)<\phi(0) \Leftrightarrow e^{-x}f(x)<0 \overset{e^{-x}>0}{\Longleftrightarrow} f(x)<0$

Άρα $\forall x\neq 0$, $x, f(x)$: ομόσημοι άρα $xf(x)>0$.

ii) $\int_0^1 f(x)\,dx<f(1)$

$\forall x\in\mathbb{R}$: $f(x)<f'(x)$ άρα και $\int_0^1 f(x)\,dx<\int_0^1 f'(x)\,dx \Leftrightarrow$

$$\Longleftrightarrow \int_0^1 f(x)dx < [f(x)]_0^1 \Leftrightarrow \int_0^1 f(x)dx < f(1)$$

13.21

$f:[0,\frac{\pi}{2}]\to\mathbb{R}$

$$\int_0^{\frac{\pi}{2}} f^2(x)dx = 4\int_0^{\frac{\pi}{2}} f(x)\eta\mu x\,dx - \pi \quad (1)$$

α) $\int_0^{\frac{\pi}{2}} 4\eta\mu^2 x\,dx = \pi$ (εκτός ύλης)

β) $(1) \Rightarrow \int_0^{\frac{\pi}{2}} f^2(x)\,dx = 4\int_0^{\frac{\pi}{2}} f(x)\eta\mu x\,dx - \int_0^{\frac{\pi}{2}} 4\eta\mu^2 x\,dx \Leftrightarrow$

$$\int_0^{\frac{\pi}{2}} \left(f^2(x) - 4f(x)\eta\mu x + 4\eta\mu^2 x\right)dx = 0 \Leftrightarrow$$

$$\int_0^{\frac{\pi}{2}} \left(f(x) - 2\eta\mu x\right)^2 dx = 0.$$

Επειδή $\left(f(x)-2\eta\mu x\right)^2 \geq 0 \ \forall\, x\in[0,\frac{\pi}{2}]$

αλλά $\int_0^{\frac{\pi}{2}} (f(x)-2\eta\mu x)^2 dx = 0$ από θεώρημα

συνεπάγεται ότι $\left(f(x)-2\eta\mu x\right)^2 = 0 \Leftrightarrow$

$f(x) - 2\eta\mu x = 0 \Leftrightarrow$

$f(x) = 2\eta\mu x, \ x\in[0,\frac{\pi}{2}]$.

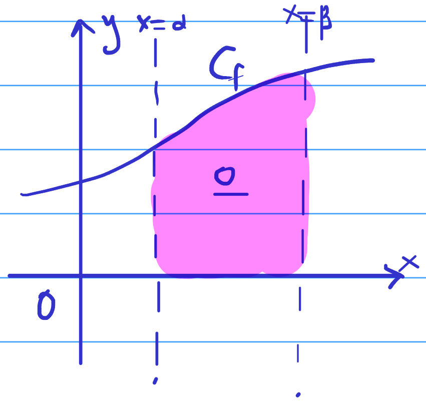
Αν $f(x)\geq 0$ και $\int_\alpha^\beta f(x)\,dx = 0$ τότε $f(x)=0 \ \forall\, x\in D_f$.

γ) $f'(x) = 2\sigma\upsilon\nu x, \ x\in[0,\frac{\pi}{2}]$

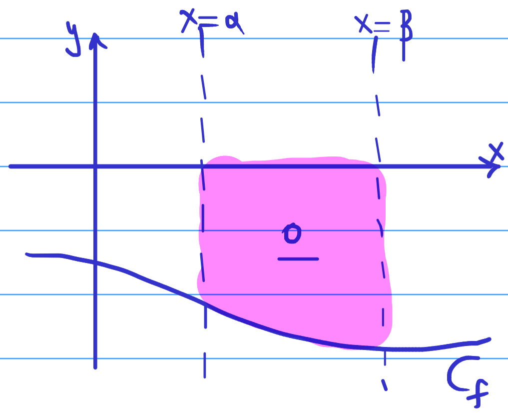
Είναι $\int_0^{\frac{\pi}{3}} \frac{f(x)}{f'(x)}dx = \int_0^{\frac{\pi}{3}} \frac{\eta\mu x}{\sigma\upsilon\nu x}dx = -\int_0^{\frac{\pi}{3}} \frac{(\sigma\upsilon\nu x)'}{\sigma\upsilon\nu x}dx = -\left[\ln(\sigma\upsilon\nu x)\right]_0^{\frac{\pi}{3}} =$

$= -\ln(\sigma\upsilon\nu\frac{\pi}{3}) + \ln(\sigma\upsilon\nu 0) = -\ln\frac{1}{2} + \ln 1 = -(-\ln 2) = \ln 2.$

Εμβαδόν μεταξύ C_f, xx', $x=\alpha$, $x=\beta$

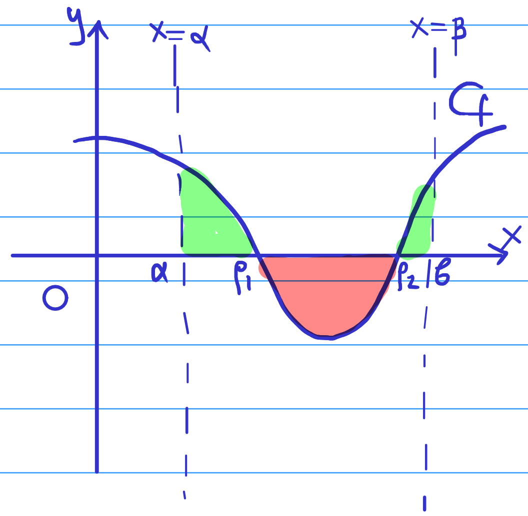


$$E(\Omega)=\int_{\alpha}^{\beta} f(x)\,dx$$



$$E(\Omega)=-\int_{\alpha}^{\beta} f(x)\,dx$$

Άρα $E(\Omega)=\int_{\alpha}^{\beta} |f(x)|\,dx$, δηλαδή:



x	α		ρ_1		ρ_2		β
$f(x)$		+	0	−	0	+	

$$E(\Omega)=\int_{\alpha}^{\beta} |f(x)|\,dx=$$

$$=\int_{\alpha}^{\rho_1} f(x)\,dx-\int_{\rho_1}^{\rho_2} f(x)\,dx+\int_{\rho_2}^{\beta} f(x)\,dx.$$

14.11

$f(x) = x^2 - 6x + 8$

Φτιάχνουμε πίνακα προσήμων της $f(x)$:

$\Delta = (-6)^2 - 4 \cdot 8 = 36 - 32 = 4 > 0$, άρα ... $x_{1,2} = \dfrac{6 \pm 2}{2} = \begin{cases} 4 \\ 2 \end{cases}$

x	$-\infty$	1	2	3	4	$+\infty$
$f(x)$	$+$	$+$	0 $-$	$-$	0 $+$	

Άρα $E(\Omega) = \int_1^3 |f(x)|dx = \int_1^2 f(x)dx - \int_2^3 f(x)dx = \left[\frac{x^3}{3} - 3x^2 + 8x\right]_1^2 - \left[\frac{x^3}{3} - 3x^2 + 8x\right]_2^3 =$

$= \frac{8}{3} - 12 + 16 - \frac{1}{3} + 3 - 8 + \frac{8}{3} - 12 + 16 - 9 + 27 - 24 =$

$= \cancel{5} + 4 - \cancel{5} + 4 + 3 = 11$ τετραγωνικές μονάδες.

14.13

$f(x) = x^3 - x$, $x \in \mathbb{R}$

$f(x) = 0 \Leftrightarrow x^3 - x = 0 \Leftrightarrow x(x^2 - 1) = 0 \Leftrightarrow x = 0$ ή $x = 1$ ή $x = -1$

x	$-\infty$	-1	0	1	$+\infty$
$f(x)$	$-$	0 $+$	0 $-$	0 $+$	

$E(\Omega) = \int_{-1}^1 |f(x)|dx = \int_{-1}^0 f(x)dx - \int_0^1 f(x)dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^0 - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 =$

$= -\cancel{\frac{1}{4}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{4}} + \frac{1}{2} = \frac{1}{2}$ τετραγωνικής μονάδας

14.15

$f(x) = 3 - 3x^2, \ x \in \mathbb{R}$

α) $f(x) = 0 \Leftrightarrow 3 - 3x^2 = 0 \Leftrightarrow x^2 = 1 \Leftrightarrow |x| = 1 \Leftrightarrow x = 1$ ή $x = -1$

x	$-\infty$		-1		1		$+\infty$
$f(x)$		$-$	0	$+$	0	$-$	

$$E(\Omega_1) = \int_{-1}^{1} f(x)dx = \left[3x - x^3\right]_{-1}^{1} = 3 - 1 - (-3+1) = 2 - (-2) = 4 \text{ τ.μ.}$$

β) C_f, $x = 2$, xx', yy'
$\uparrow$
$x = 0$

x	$-\infty$		-1		0		1		2		$+\infty$
$f(x)$		$-$	0	$+$		$+$	0	$-$		$-$	

Άρα $E(\Omega_2) = \int_0^1 f(x)dx - \int_1^2 f(x)dx = \left[3x - x^3\right]_0^1 - \left[3x - x^3\right]_1^2 =$

$= \cancel{3} - 1 - \cancel{6} + 8 + \cancel{3} - 1 = 6$ τ.μ.

14.17

$f(x) = e^x + 2x - 1, \ x \in \mathbb{R}$

α) ... $f'(x) = e^x + 2 > 0 \ \forall x \in \mathbb{R} \Longrightarrow f: \uparrow \mathbb{R}$ με $f(0) = e^0 - 1 = 0$.

$\nearrow f: 1-1$ ως γν. μον.

Άρα $f(x) = 0 \Leftrightarrow f(x) = f(0) \overset{f:1-1}{\Longleftrightarrow} x = 0$.

β) • $x > 0 \overset{f:\uparrow}{\Longleftrightarrow} f(x) > f(0) \Leftrightarrow f(x) > 0$

• $x < 0 \overset{f:\uparrow}{\Longleftrightarrow} f(x) < f(0) \Leftrightarrow f(x) < 0$

x	$-\infty$		-1		0		1		$+\infty$
$f(x)$		$-$		$-$	0	$+$		$+$	

Άρα $E(\Omega) = -\int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx = -\left[e^x + x^2 - x\right]_{-1}^{0} + \left[e^x + x^2 - x\right]_{0}^{1} =$

$= \cancel{-1} + \frac{1}{e} \cancel{+1} \cancel{+1} + e \cancel{+1} \cancel{-1} \cancel{-1} = \frac{1}{e} + e = \frac{e^2+1}{e}$ τ.μ.