14/01/2026

$f'(x) = f(x) \ \forall x \in \Delta \Longrightarrow \exists c \in \mathbb{R} : f(x) = ce^x$. Δεν το χρησιμοποιώ έτοιμο!

1.59

$f : (0, +\infty) \to \mathbb{R}$, $f\left(\frac{1}{2}\right) = 2\sqrt{e}$

$xf'(x) = (x-1)f(x) \ \forall x > 0$

α) τύπος της f

$\forall x > 0$:

$$xf'(x) = (x-1)f(x) \Longrightarrow$$
$$xf'(x) = xf(x) - f(x) \Longrightarrow$$
$$xf'(x) + f(x) = xf(x) \Longrightarrow$$
$$[xf(x)]' = xf(x) \Longrightarrow$$
$$[xf(x)]' - xf(x) = 0 \Longrightarrow$$
$$[xf(x)]'e^{-x} - xf(x)e^{-x} = 0 \Longrightarrow$$
$$[xf(x)]'e^{-x} + xf(x)(e^{-x})' = 0 \Longrightarrow$$
$$\left(xf(x)e^{-x}\right)' = 0$$

άρα από θεώρημα $\exists c \in \mathbb{R}$: $\frac{xf(x)}{e^x} = c \Rightarrow xf(x) = ce^x$

Για $x = \frac{1}{2}$: $\frac{1}{2}f\left(\frac{1}{2}\right) = c\sqrt{e} \Longrightarrow \frac{1}{2} \cdot 2\sqrt{e} = c\sqrt{e} \Rightarrow c = 1$,

άρα $\forall x > 0$: $xf(x) = e^x \Longrightarrow f(x) = \frac{e^x}{x}$.

β) $f'(x) = \frac{e^x \cdot x - e^x}{x^2} = \frac{e^x}{x} - \frac{e^x}{x^2}$, $x > 0$

$$f''(x) = \frac{e^x \cdot x - e^x}{x^2} - \frac{e^x \cdot x^2 - e^x \cdot 2x}{x^4} =$$

$$= \frac{e^x \cdot x - e^x}{x^2} - \frac{e^x \cdot x - e^x \cdot 2}{x^3} =$$

$$= \frac{e^x \cdot x^2 - e^x \cdot x - e^x \cdot x + 2e^x}{x^3} = \frac{e^x(x^2 - 2x + 2)}{x^3}$$

$$\lim_{x\to+\infty}\frac{f''(x)}{f'(x)}=\lim_{x\to+\infty}\frac{\frac{e^x(x^2-2x+2)}{x^3}}{\frac{e^x(x-1)}{x^2}}=\lim_{x\to+\infty}\frac{x^2-2x+2}{x^2-x}=\cdots=1$$

1.64

$f''(x)=f(x)\ \forall\, x\in\mathbb{R}$

εφαπτ. C_f στο $A(0,f(0))$: $y=3x-1$

α) $f(0),\ f'(0)=;$

$A(0,f(0))$: $y-f(0)=f'(0)(x-0)\Longrightarrow y-f(0)=f'(0)\cdot x\Rightarrow y=f'(0)\cdot x+f(0)$

Άρα ισχύει $\begin{cases} f'(0)=3 \\ f(0)=-1 \end{cases}$.

β) τύπος της f

$\forall\, x\in\mathbb{R}$: $f''(x)=f(x)\Longrightarrow$

$f''(x)+f'(x)=f'(x)+f(x)\Longrightarrow$

$[f'(x)+f(x)]'=f'(x)+f(x)\Longrightarrow$

$[f'(x)+f(x)]'-(f'(x)+f(x))=0\Longrightarrow$

$[f'(x)+f(x)]'e^{-x}-(f'(x)+f(x))e^{-x}=0\Longrightarrow$

$[(f'(x)+f(x))e^{-x}]'=0$ άρα από θεώρημα

$\exists\, c_1\in\mathbb{R}$: $\dfrac{f'(x)+f(x)}{e^x}=c_1\Rightarrow f'(x)+f(x)=c_1e^x$

Για $x=0$: $f'(0)+f(0)=c_1\Longrightarrow c_1=2$, άρα

$\forall\, x\in\mathbb{R}$: $f'(x)+f(x)=2e^x\Longrightarrow$

$f'(x)e^x+f(x)e^x=2e^{2x}\Longrightarrow$

$[e^xf(x)]'=[e^{2x}]'$ άρα από θεώρημα

$\exists\, c_2\in\mathbb{R}$: $e^xf(x)=e^{2x}+c_2\ \forall\, x\in\mathbb{R}$

Για $x=0$: $f(0)=1+c_2\Longrightarrow c_2=-2$

άρα $\forall\, x\in\mathbb{R}$: $e^xf(x)=e^{2x}-2\Longrightarrow f(x)=e^x-\dfrac{2}{e^x}$.

$f' > 0 \ \forall x \in \Delta \Longrightarrow f: \uparrow \Delta$

$f' < 0 \ \forall x \in \Delta \Longrightarrow f: \downarrow \Delta$

γ) $f(\mathbb{R}) = ;$

$$f'(x) = \left(e^x - \frac{2}{e^x}\right)' = \left(e^x - 2e^{-x}\right)' = e^x + 2e^{-x} > 0 \ \forall x \in \mathbb{R}$$

άρα $f: \uparrow \mathbb{R}$ οπότε $f(\mathbb{R}) = \left(\lim\limits_{x \to -\infty} f(x), \lim\limits_{x \to +\infty} f(x)\right) = (-\infty, +\infty) = \mathbb{R}$.