23/01/2026

2.68

$$f(x) = (\lambda-1)x^3+6x^2+3(\lambda+2)x-7, \quad x\in\mathbb{R}$$

$$\dots \quad f'(x) = 3(\lambda-1)x^2+12x+3(\lambda+2), \quad x\in\mathbb{R}$$

$$f: \uparrow \mathbb{R} \Longrightarrow f'(x) \geq 0 \quad \forall\, x\in\mathbb{R}$$

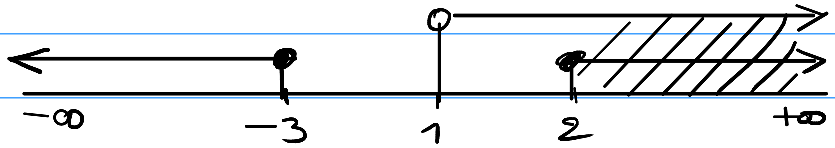
$$\forall\, x\in\mathbb{R}: \quad 3(\lambda-1)x^2+12x+3(\lambda+2) \geq 0$$

Πρέπει $\begin{cases} \Delta \leq 0 \\ \alpha > 0 \end{cases}$

- $\Delta \leq 0 \Longrightarrow 12^2 - 4\cdot 3(\lambda-1)\cdot 3(\lambda+2) \leq 0 \Longrightarrow$
  $144 - 36(\lambda^2+\lambda-2) \leq 0 \Longrightarrow$
  $4 - \lambda^2 - \lambda + 2 \leq 0 \Longrightarrow$
  $\lambda^2+\lambda-6 \geq 0 \Longrightarrow$
  $\lambda \leq -3$ ή $\lambda \geq 2$

λ	$-\infty$		-3		2		$+\infty$
$\lambda^2+\lambda-6$		+	0	−	0	+	

- $\alpha > 0 \Longrightarrow 3(\lambda-1) > 0 \Longrightarrow \lambda > 1$



Άρα $f: \uparrow \mathbb{R} \Longrightarrow \lambda \geq 2$.

2.71

$$f(x) = \frac{\alpha x^2+\beta}{x^2+3}, \quad \beta > 3\alpha, \quad \alpha \neq 0, \quad f(\mathbb{R}) = (\beta-7, 2\alpha-1]$$

$$\dots \quad f'(x) = \frac{2\alpha x(x^2+3) - (\alpha x^2+\beta)\cdot 2x}{(x^2+3)^2} = \frac{2x(\cancel{\alpha x^2} + 3\alpha - \cancel{\alpha x^2} - \beta)}{(x^2+3)^2} =$$

$$= \frac{2x(3\alpha-\beta)}{(x^2+3)^2}$$

x	$-\infty$		0		$+\infty$
$2x$		$-$	0	$+$	
$3\alpha-\beta$		$-$		$-$	
f'		$+$	0	$-$	
f	α	↗	Ο.Μ. $f(0)=\frac{\beta}{3}$	↘	α

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\alpha x^2+\beta}{x^2+3} \overset{\alpha\neq 0}{=\!=} \lim_{x\to-\infty} \frac{\alpha \cancel{x^2}}{\cancel{x^2}} = \alpha \left(= \lim_{x\to+\infty} f(x)\right)$$

Άρα $f(\mathbb{R}) = \left(\alpha, \frac{\beta}{3}\right]$. Όμως δίδεται $f(\mathbb{R}) = (\beta-7, 2\alpha-1]$.

Πρέπει $\begin{cases} \alpha = \beta - 7 \\ \frac{\beta}{3} = 2\alpha - 1 \end{cases}$

Είναι $\frac{\beta}{3} = 2\beta - 14 - 1 \Longrightarrow \beta = 6\beta - 45 \Longrightarrow -5\beta = -45 \Longrightarrow$
$\Longrightarrow \boxed{\beta = 9}$ και $\alpha = 9 - 7 = 2 \Longrightarrow \boxed{\alpha = 2}$

2.78

$f(x) = \frac{e}{x} - \ln x, \ x>0$

α) ... $f'(x) = -\frac{e}{x^2} - \frac{1}{x} < 0 \ \forall\, x>0$ άρα $f: \downarrow (0,+\infty)$

β) $f: \downarrow (0,+\infty) \Longrightarrow f: 1-1$ ως γν. μον.

i) $\frac{e}{|x|+3} - \frac{e}{2|x|+1} = \ln\frac{|x|+3}{2|x|+1} \Longrightarrow$

$\frac{e}{|x|+3} - \ln(|x|+3) = \frac{e}{2|x|+1} - \ln(2|x|+1) \Longrightarrow$

$$f(|x|+3)=f(2|x|+1) \xRightarrow{f:1-1}$$
$$|x|+3=2|x|+1 \Rightarrow$$
$$|x|=2 \Rightarrow$$
$$x=2 \text{ ή } x=-2.$$

ii) $(e-x\ln x)^2=e^2x-ex^2\ln x \xRightarrow{x>0}$
$$(e-x\ln x)^2=ex(e-x\ln x) \Rightarrow$$
$$(e-x\ln x)^2-ex(e-x\ln x)=0 \Rightarrow$$
$$(e-x\ln x)(e-x\ln x-ex)=0 \Rightarrow$$
$$e-x\ln x=0 \text{ ή } e-x\ln x-ex=0 \Rightarrow$$
$$\frac{e}{x}-\ln x=0 \text{ ή } \frac{e}{x}-\ln x=e \Rightarrow$$
$$f(x)=f(e) \text{ ή } f(x)=f(1) \xRightarrow{f:1-1}$$
$$x=e \text{ ή } x=1.$$

2.84

$$f(x)=\frac{x}{x-2}-\ln(x-1)\ ,\quad D_f=(1,2)\cup(2,+\infty)$$

a) ... $f'(x)=\frac{\cancel{x}-2-\cancel{x}}{(x-2)^2}-\frac{1}{x-1}=-\frac{2}{(x-2)^2}-\frac{1}{x-1}<0$

$\forall\ 1<x\neq 2$, άρα $f:\downarrow (1,2)$ και $f:\downarrow(2,+\infty)$.

β)

x	1		2		$+\infty$
f'		$-$	‖	$-$	
f	$+\infty$	$\searrow$	$-\infty$ ‖ $+\infty$	$\searrow$	$-\infty$

$$\lim_{x\to1^+}f(x)=\lim_{x\to1^+}\left(\frac{x}{x-2}-\ln(x-1)\right)\overset{-1-(-\infty)}{=\!=\!=}+\infty$$

$$\lim_{x\to2^-}f(x)=\lim_{x\to2^-}\left(x\cdot\frac{1}{x-2}-\ln(x-1)\right)\overset{(+2)(-\infty)}{=\!=\!=}-\infty$$

$\lim\limits_{x\to2^+}f(x)=\ldots=+\infty$ / $\lim\limits_{x\to+\infty}f(x)=\lim\limits_{x\to+\infty}\left(\frac{x}{x-2}-\ln(x-1)\right)\overset{+1-(+\infty)}{=\!=\!=}-\infty$

$$e^{\frac{x}{x-2}} = x-1 \xRightarrow[x\neq 2]{x>1} \frac{x}{x-2} = \ln(x-1) \Longrightarrow f(x)=0 \quad \ldots \text{ ακρ. } 2 \text{ ρίζες}$$

$f: \downarrow (1,2)$ με $f((1,2)) = (-\infty,+\infty) = \mathbb{R}$ άρα $\exists$ ακρ. 1 $\xi_1 \in (1,2): f(\xi_1)=0$

$f: \downarrow (2,+\infty)$ με $f((2,+\infty)) = (-\infty,+\infty) = \mathbb{R}$ άρα $\exists$ ακρ. 1 $\xi_2 \in (2,+\infty): f(\xi_2)=0$

Επειδή $(1,2) \cap (2,+\infty) = \emptyset$ ...

2.87

$f: \mathbb{R} \to \mathbb{R}$, $f(0)=1$

$2x f(x) + (x^2+2) f'(x) = 4 \quad \forall\, x \in \mathbb{R} \quad (1)$

α) $(1) \Longrightarrow [(x^2+2)f(x)]' = (4x)'$ άρα από θεώρημα $\exists\, c \in \mathbb{R}$:

$(x^2+2)f(x) = 4x + c \quad \forall\, x \in \mathbb{R}$.

Για $x=0$: $2f(0) = c \Longrightarrow c = 2$

Άρα $\forall\, x \in \mathbb{R}$: $(x^2+2)f(x) = 4x+2 \Longrightarrow f(x) = \dfrac{4x+2}{x^2+2}$.

β) ... $f'(x) = \dfrac{4(x^2+2) - (4x+2)\cdot 2x}{(x^2+2)^2} = \dfrac{4x^2+8-8x^2-4x}{(x^2+2)^2} =$

$$= \frac{-4x^2-4x+8}{(x^2+2)^2}, \quad x \in \mathbb{R}.$$

$f'(x)=0 \Longrightarrow -4x^2-4x+8=0 \Longrightarrow x^2+x-2=0 \Longrightarrow x=-2$ ή $x=1$.

x	$-\infty$		-2		1		$+\infty$
f'		$-$	0	$+$	0	$-$	
f	0	$\searrow$	-1	$\nearrow$	2	$\searrow$	0

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4\cancel{x}}{x^{\cancel{2}}} = 0 = \lim_{x\to+\infty} f(x)$$

$$f(-2) = \frac{-8+2}{6} = -1$$

$$f(1) = \frac{6}{3} = 2$$

$$f(x) = \frac{4x+2}{x^2+2}$$

$f(\mathbb{R}) = [-1, 2]$.

γ) $(\eta\mu x-3)(x^2+2)=4x+2 \Longrightarrow$
$f(x)=\eta\mu x-3$.

$\forall\, x\in\mathbb{R}:\ -1\le \eta\mu x\le 1 \Longrightarrow$
$-4\le \eta\mu x-3\le -2$

Άρα $(\eta\mu x-3)\in[-4,-2]\ \forall x\in\mathbb{R}$

ενώ $f(x)\in[-1,2]\ \forall\, x\in\mathbb{R}$

Επειδή $[-4,-2]\cap[-1,2]=\emptyset$

$\nexists\, x_0\in\mathbb{R}:\ f(x_0)=\eta\mu x_0-3$ άρα...

2.92 Πλήθος ριζών

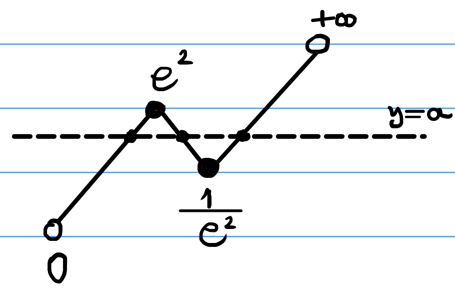
$$e^{x^3+2}=\alpha e^{3x^2}\quad \forall\ \alpha\in\mathbb{R}\quad (1)$$

$$(1)\Longrightarrow \underbrace{e^{x^3-3x^2+2}}_{f(x)}=\alpha\ ,\ \alpha\in\mathbb{R}$$

$$\ldots\ f'(x)=e^{x^3-3x^2+2}\cdot(3x^2-6x),\quad x\in\mathbb{R}$$

x	$-\infty$		0		2		$+\infty$
f'		$+$	0	$-$	0	$+$	
f	0	$\nearrow$	e^2	$\searrow$	$\frac{1}{e^2}$	$\nearrow$	$+\infty$

$f(x)=e^{x^3-3x^2+2}$



$f(x)=\alpha\ ,\ \alpha\in\mathbb{R}\quad (1)$

- Αν $\alpha\le 0$, τότε (1): αδύνατη
- Αν $0<\alpha<\frac{1}{e^2}$, τότε (1): 1 λύση
- Αν $\alpha=\frac{1}{e^2}$, τότε (1): 2 λύσεις
- Αν $\frac{1}{e^2}<\alpha<e^2$, τότε (1): 3 λύσεις
- Αν $\alpha=e^2$, τότε (1): 2 λύσεις
- Αν $\alpha>e^2$, τότε (1): 1 λύση.